Sec 5.2 Higher order linear ODEs

<u>2nd order lin ODE</u> $\quad y'' + p(x)y' + q(x)y = f(x)$

<u>Associated homog. eqn:</u> $\quad y'' + p(x)y' + q(x)y = 0$

<u>Thm</u> General solutions for (any)

① $$y'' + p(x)y' + q(x)y = 0$$

are lin. combs.

$$y = c_1\boxed{y_1} + c_2\boxed{y_2}$$

y_1, y_2 must be lin. indep funcs.

("solution set for ODE ① is <u>2-dimensional</u>")

<u>Thm</u> General solutions for nonhomogeneous

$$y'' + p(x)y' + q(x)y = f(x)$$

are $y = \boxed{c_1\boxed{y_1} + c_2\boxed{y_2}} + \boxed{y_p}$

this is gen solution for assoc. homog. eqn

$$y'' + py' + qy = 0$$

this "particular sol" accounts for $f(x)$ being $\neq 0$.

- functions f, g (or y_1, y_2) are <u>lin indep</u>

$\iff c_1 f + c_2 g = 0$ (const) possible <u>only</u> w/ $(c_1, c_2) = (0, 0)$

$\iff W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \neq 0$ everywhere (on x interval/domain)

Higher-order

(shown only for 3^{rd} order, but pattern continues for $4^{th}, 5^{th}, \dots n^{th}, \dots$ order)

General linear ODE

$$y^{(3)} + p_2(x)\,y'' + p_1(x)y' + p_0(x)y = f(x)$$

Assoc homog eqn

$$y^{(3)} + p_2(x)y'' + p_1(x)y' + p_0(x)\,y = 0$$

general solutions for homog. eqn are

$$y = c_1\boxed{y_1} + c_2\boxed{y_2} + c_3\boxed{y_3}$$

y_1, y_2, y_3 must be lin indep. on domain/interval

general solutions for nonhomog. eq are

$$y = \underbrace{\boxed{c_1y_1 + c_2y_2 + c_3y_3}}_{\text{gen solution for associated homog. eqn.}} + \underbrace{\boxed{y_p}}_{\text{particular solution to account for nonhomog. piece}}$$

gen solution for associated homog. eqn.

particular solution to account for nonhomog. piece

Independence

y_1, y_2, y_3 lin indep on interval if lin comb

$c_1y_1 + c_2y_2 + c_3y_3 \equiv 0$ possible only with $(c_1, c_2, c_3) = (0,0,0)$

Otherwise, lin depen.

Check indep.: What (c_1, c_2, c_3) make

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0 \quad ?$$

(deriv.) $\Longrightarrow$ $c_1 y_1' + c_2 y_2' + c_3 y_3' = 0$

(deriv.) $\Longrightarrow$ $c_1 y_1'' + c_2 y_2'' + c_3 y_3'' = 0$

$$\Rightarrow \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

only trivial (c_1, c_2, c_3) $\Longleftrightarrow$ determinant

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0 \text{ everywhere}$$

(If $W(y_1, y_2, y_3) = 0 \Rightarrow$ dependent)

Ex $\quad y_1 = 1 \quad y_2 = \cos^2(x) \quad y_3 = \sin^2(x)$

$W(1, \cos^2 x, \sin^2 x) = \cdots = 0$, so lin depen.

By "inspection"

$$\cos^2 x + \sin^2 x = 1$$

$\Rightarrow$ $(-1)\,1 + (1)\cos^2(x) + (1)\sin^2 x = 0$

$(-1, 1, 1) \neq (0, 0, 0) \Rightarrow$ depen too.

Wronskian for 4 functions $\longrightarrow$ 4×4 (bottom row has y_i''')

<u>Examples/thm</u> (just <u>some</u>)

- $\{1, x, x^2, x^3, \dots\}$ all lin indep
- $\{e^{ax}, e^{bx}\}$ lin indep if $a \neq b$
- $\{\cos, \sin\}$ lin indep
- $\{\cos(ax), \cos(bx)\}$ indep if $a \neq b$
- $\{\sin(ax), \sin(bx)\}$ " " " $(a, b \neq 0)$
- $\{\cos, \cos^2, \cos^3, \dots\}$ indep
- ★ if f is not 0-function, then $f, x\cdot f, x^2 f, \dots$ all lin indep

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<u>Reduction of order</u>

sometimes can "bootstrap" solutions by reducing order

"Euler equation" $x^2 y'' + x y' - 9y = 0$

$\rightarrow y'' + x^{-1} y' - 9x^{-2} y = 0$

suppose known solution is $y_1 = x^3$.

2^{nd} order $\Rightarrow$ need y_2, linearly indep from y_1

(Can't use $k y_1 = k x^3$)

Try nonconst $v(x) \longrightarrow y_2 = v(x) y_1 = x^3 v(x)$

$$\begin{cases} y_2 = x^3 v \\ y_2' = 3x^2 v + x^3 v' \\ y_2'' = 6xv + 6x^2 v' + x^3 v'' \end{cases}$$

sub into $y'' + x^{-1}y' - 9x^{-2}y = 0$, ... (simplify)...

result is $v'' + 7x^{-1}v' = 0$

$w = v'$, $w' = v'' = dw/dx$

$$\frac{dw}{dx} + 7w/x = 0$$

$$\Rightarrow \frac{dw}{w} = -7\frac{dx}{x} \quad \text{(separable)}$$

...

$$v' = w = Cx^{-7}$$

$$\Rightarrow \boxed{v = Cx^{-6} + D}$$

So using gives

$$y_2 = v \cdot x^3 = Cx^{-3} + Dx^3$$

since we want to avoid/ be indep of $ky_1 = kx^3$,

easiest to use $C = 1$, $D = 0$

$$\Rightarrow \boxed{y_2 = x^{-3}.}$$

Total general solution $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 x^3 + c_2 x^{-3}}$$